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NORMED BARELY BAIRE SPACES

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ABSTRACT
We construct two prehilbertian Baire spaces whose product is not a Baire space.

Although Oxtoby [4] and Aarts and Lutzer [1] have proved that the product of two Baire spaces is a Baire space under some mild additional conditions, it is well-known that, in the general case, this result does not hold.

Using the continuum hypothesis, Oxtoby [4] has given the first example of a completely regular Baire space whose square is not a Baire space. More recently some examples of barely Baire spaces have been given without additional hypotheses. See [2], where there is a review of the history of the problem.

All these examples are metric spaces but it was an open question if there are Baire locally convex vector spaces whose product is not a Baire space. For the related class of unordered Baire-like spaces, Todd and Saxon [5] have proved that every arbitrary product of unordered Baire-like spaces is unordered Baire-like.

In this paper, using the technique of Fleissner and Kunen [2, example 1], we derive an example of two prehilbertian Baire spaces whose product is not a Baire space.

We shall denote by $l^2(\omega_1)$ the Hilbert space of all scalar sequences $\langle x_\alpha : \alpha < \omega_1\rangle$ such that $\Sigma_{\alpha<\omega_1} |x_\alpha|^2 < +\infty$.

For every $x = \langle x_\alpha : \alpha < \omega_1\rangle$ in $l^2(\omega_1)$, only countably many coordinates do not vanish, so that we can define $f : l^2(\omega_1) \to \omega_1$ by

$$f(x) = \sup\{\alpha < \omega_1 : x_\alpha \neq 0\}.$$

Also for every natural number $n \geqq 1$ we shall denote

$$f_n(x) = \sup\{\alpha < \omega_1 : |x_\alpha| \geqq 1/n\}.$$

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Then, for every $x \in l^2(\omega_1)$, it is clear that

$$f(x) = \sup f_n(x).$$

These functions verify the following conditions too:

(a) If $\alpha < f(x)$, there exists a neighborhood V of x such that z in V implies $f(z) > \alpha$.

(b) For every $x \in l^2(\omega_1)$ and every natural number $n \geqq 1$, there exists a neighborhood V_n of x such that z in V_n implies $f_n(z) \leqq f_n(x)$.

PROOF. (a) If $\alpha < f(x)$, then there exists β, $\alpha < \beta \leqq f(x)$, verifying $x_\beta \neq 0$. There is a neighborhood V of x such that z in V implies $z_\beta \neq 0$ and then z in V implies $f(z) \geqq \beta > \alpha$.

(b) If $x \in l^2(\omega_1)$, then $\sup\{|x_\alpha| : \alpha < \omega_1 \text{ and } |x_\alpha| < 1/n\} < 1/n$, since $x_\alpha \to 0$. Therefore we can choose $r > 0$ such that $\|x - y\| < r$ implies $|y_\alpha| < 1/n$ whenever $|x_\alpha| < 1/n$. So $V_n = \{y : \|x - y\| < r\}$ satisfies (b).

We shall denote by $\mathcal{B}$ the base for the topology of $l^2(\omega_1)$ consisting of the open balls of rational radius and center $\langle x_\alpha : \alpha < \omega_1 \rangle$ such that $\{\alpha \in \omega_1 : |x_\alpha| \neq 0\}$ is finite and every x_α is rational.

For every $\gamma < \omega_1$, let $\mathcal{B}_\gamma$ be the subset of $\mathcal{B}$ of all balls with center $\langle x_\alpha : \alpha < \omega_1 \rangle$ such that $x_\alpha = 0$ if $\alpha \geqq \gamma$.

Before defining our space we need a lemma.

LEMMA 1. *There exists a dense linear subspace M of l^2 that is a Baire space and does not contain any finite linear combination of basic vectors.*

PROOF. Let $\langle a_\alpha : \alpha < \omega^2 \rangle$ be a sequence of linearly independent vectors in $L^2[0,1]$ such that, for every $n < \omega$, $\langle a_{\omega n + k} : k < \omega \rangle$ is an orthonormal basis of $L^2[0,1]$. For example, let $\langle a_{\omega n + k} : k < \omega \rangle$ be the only orthonormal basis such that $a_{\omega n + k}$ is the class determined by the function $e^{nx} p_k(x)$, where $p_k(x)$ is a polynomial of degree k.

We complete $\langle a_\alpha : \alpha < \omega^2 \rangle$ to construct a Hamel basis $\langle a_\alpha : \alpha < \Omega \rangle$ of $L^2[0,1]$. Let V_n $(n < \omega)$ be the linear space spanned by the vectors a_α that are not contained in the orthonormal basis $\langle a_{\omega n + k} : k < \omega \rangle$. It is easy to see that every V_n is dense in $L^2[0,1]$ and that $L^2[0,1] = \bigcup \{V_n : n < \omega\}$.

Since $L^2[0,1]$ is a Baire space, we get that there exists a natural number q such that V_q is of second category in $L^2[0,1]$. Then V_q, being a dense linear subspace of second category in $L^2[0,1]$, is a Baire space.

Finally let $T : L^2[0,1] \to l^2$ be the bounded linear operator defined by $T(\sum_{k<\omega} x_k a_{\omega q + k}) = \langle x_k : k < \omega \rangle$. It is clear that $M = T(V_q)$ satisfies the lemma.

For every limit ordinal γ, $0<\gamma<\omega_1$, we fix an increasing sequence of ordinals $\langle\alpha_n : n<\omega\rangle$ verifying that $\sup\alpha_n=\gamma$ and $\gamma\setminus\{\alpha_n : n<\omega\}$ is infinite.

For every limit ordinal γ, $0<\gamma<\omega_1$, we denote by H_γ the closed linear subspace of $l^2(\omega_1)$ formed by all sequences $\langle x_\alpha : \alpha<\omega_1\rangle$ such that $x_\alpha=0$ whenever $\alpha\geqq\gamma$ and we denote by M_γ the dense linear subspace of all sequences $\langle x_\alpha : \alpha<\omega_1\rangle\in H_\gamma$ satisfying $\langle x_{\alpha_n} : n<\omega\rangle\in M$, where M is the space defined in Lemma 1 and $\langle\alpha_n : n<\omega\rangle$ the sequence associated with γ.

Since M_γ is isomorphic to $M\times l^2$, we get from the Oxtoby theorem [4] that M_γ is a Baire space.

For every $A\subset\omega_1$ consisting of limit ordinals γ, $\gamma\neq0$, let A^* be the prehilbertian linear subspace of $l^2(\omega_1)$ spanned by $\bigcup\{M_\gamma : \gamma\in A\}$.

It is clear that M_γ is contained in A^* for every $\gamma\in A$. Furthermore, for every $x\in A^*$, there is a finite sequence in A, $\gamma_1<\gamma_2<\cdots<\gamma_n$, verifying $x=\sum_{i=1}^n x_i$ where $x_i\in M_{\gamma_i}$. Clearly $f(x)=\gamma_n$ by the definition of M_γ and Lemma 1. Thus $f(x)$ is in A if x is in A^*.

LEMMA 2. *If A is stationary, then A^* is a Baire space.*

PROOF. If A is stationary, $\sup A=\omega_1$, so that A^* is dense in $l^2(\omega_1)$. Every dense open subset of A^* is the intersection of a dense open subset D of $l^2(\omega_1)$ and A^*. Hence it suffices to prove that $(\bigcap D_n)\cap G\cap A^*\neq\emptyset$ for every sequence of open dense sets D_n in $l^2(\omega_1)$ and every nonempty open set G in $l^2(\omega_1)$.

Every open dense set D in $l^2(\omega_1)$ induces a function $\tilde{D} : \mathcal{B}\to\mathcal{B}$ such that, for every $B\in\mathcal{B}$, we have $\tilde{D}(B)\subset B\cap D$. Since every $\mathcal{B}_\alpha$ is countable, there is $\hat{D}(\alpha)<\omega_1$ such that $\tilde{D}(B)$ belongs to $\mathcal{B}_{\hat{D}(\alpha)}$ for every $B\in\mathcal{B}_\alpha$.

Let $\mathcal{A}$ be the set of the functions $\hat{D}_n$. By Kunen [3, lemma II.6.13], the set $C=\{\gamma<\omega_1 : \gamma$ is closed under $\mathcal{A}\}$ is a closed unbounded set of ω_1.

Let B be a subset of G, $B\in\mathcal{B}_\alpha$. Pick $\gamma>\alpha$ such that $\gamma\in C\cap A$. This choice is possible since A is stationary.

Since γ is closed under $\hat{D}_n$, we have that $D_n\cap H_\gamma$ is an open dense set in H_γ. Since $\gamma>\alpha$, $G\cap H_\gamma$ is a nonempty open set of H_γ. Since M_γ is dense in H_γ and M_γ is a Baire space, we get that $(\bigcap D_n)\cap G\cap M_\gamma\neq\emptyset$.

LEMMA 3. *If A_0, A_1 are disjoint stationary sets of ω_1, then $A_0^*\times A_1^*$ is not a Baire space.*

PROOF. Define

$$D_n=\{(x,y)\in l^2(\omega_1)\times l^2(\omega_1) : \min(f(x),f(y))>\max(f_n(x),f_n(y))\}.$$

D_n is open: Let $(x, y) \in D_n$. There exist neighborhoods V_x, V_y of x and y respectively satisfying that if (u, v) belongs to $V_x \times V_y$, then $f(u) > \max(f_n(x), f_n(y))$, $f(v) > \max(f_n(x), f_n(y))$, $f_n(u) \leqq f_n(x)$, and $f_n(v) \leqq f_n(y)$, so that if $(u, v) \in V_x \times V_y$, then $\min(f(u), f(v)) > \max(f_n(u), f_n(v))$.

D_n is dense: For $(x, y) \in l^2(\omega_1) \times l^2(\omega_1)$ choose α such that $\max(f(x), f(y)) < \alpha < \omega_1$. Denote by e_α the sequence $\langle \delta_\beta : \beta < \omega_1 \rangle \in l^2(\omega_1)$ where $\delta_\alpha = 1$ and $\delta_\beta = 0$ if $\beta \neq \alpha$. Then, for every t, $0 < t < 1/n$, we have $(te_\alpha + x, te_\alpha + y)$ belongs to D_n.

Finally we shall prove $(\bigcap D_n) \cap (A_0^* \times A_1^*) = \varnothing$. If (x, y) is in $A_0^* \times A_1^*$, then $f(x)$ is in A_0 and $f(y)$ is in A_1. Since $A_0 \cap A_1 = \varnothing$, $f(x) \neq f(y)$. There is no loss of generality to assume that $f(x) > f(y)$, thus there is a natural number $n \geqq 1$ satisfying $f_n(x) > f(y)$. Hence $(x, y) \notin D_n$.

THEOREM 1. *There is a family $\{A_\alpha^* : \alpha < \omega_1\}$ of different subspaces of $l^2(\omega_1)$ which are Baire spaces and such that $A_\alpha^* \times A_\beta^*$ is not a Baire space whenever $\alpha \neq \beta$.*

The proof follows from the lemmas and the classical result of Ulam (see Kunen [3, theorem II.6.11]) that proves that there are ω_1 stationary disjoint subsets of ω_1.

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